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Quasi-periodic solutions of derivative nonlinear Schrödinger equations with a given potential ☆

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ABSTRACT

It is proved that for a prescribed potential $V(x)$ there are many quasi-periodic solutions of derivative nonlinear Schrödinger equation $\mathrm{i}u_t = u_{xx} - V(x)u + |u_x|^2 u$ subject to Dirichlet boundary condition by means of a KAM theorem to a reversible system.

1. Introduction and main results

Over the recent years, there has been a substantial amount of research on the nonlinear Schrödinger equation

$$\mathrm{i}u_t = u_{xx} - V(x)u + F(x, u, \bar{u}, u_x, \bar{u}_x) \tag{1}$$

and systems of those. Here $\bar{u}$ means the complex conjugate of u. And with respect to F we can divide the system into two cases: bounded and unbounded. In order to understand the difference between the two cases we consider an abstract evolution equation

$$\frac{dw}{dt} = Aw + F(w), \tag{2}$$

where w is in some Hilbert space, say, Sobolev space H^p, and where $A : H^p \to H^{p-d}$ is a linear operator and F is a nonlinear map sending some neighborhood of H^p to $H^{p-\delta}$. One calls $d \geqslant 1$ and $\delta \in \mathbb{R}$ the orders of A and F, respectively.

(i) If $\delta \leqslant 0$, F is called a bounded perturbation.
(ii) If $\delta > 0$, F is called an unbounded perturbation.

Thus if $F(x, u, \bar{u}, u_x, \bar{u}_x)$ is independent of u_x and $\bar{u}_x$, that is, $F = F(x, u, \bar{u})$, then F is a bounded perturbation where $d = 2$ and $\delta = 0$. If $F(x, u, \bar{u}, u_x, \bar{u}_x)$ depends on u_x and $\bar{u}_x$, then (1) is called the derivative nonlinear Schrödinger equation (DNLS). In this case, $d = 2$ and $\delta = 1$, so F is an unbounded perturbation.

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When F is a bounded perturbation, there are many results for the system (1). Such as in [1], Kuksin has constructed the time quasi-periodic solutions of (1) when $V(x) = V(x;a)$ depends on a parameter a from a bounded n-dimensional domain and $V(x) = V_\omega(x)$ is a random process with the probability space. In 1996, Kuksin and Pöschel [17] investigated the case $V(x) \equiv m \in \mathbb{R}$ which was a prescribed (not random) potential. Afterward, Yuan [19] and Du, Yuan [26] generalized the conclusion into the wave equation and the Schrödinger equation respectively with a given potential $V(x)$ which was not necessary to be constant. For the existence of KAM tori of PDEs with bounded perturbations there are too many references to list here, we give just two survey papers by Kuksin [6] and Bourgain [7] again.

Then whether or not the same conclusion we can get when F is an unbounded perturbation? To this end we should acquaint ourselves with the KAM theorem for unbounded perturbation. The first KAM theorem for unbounded perturbation is due to Kuksin [2,3] where it is assumed that $d-1 > \delta$. Kuksin's theorem in [3] is used to prove the persistence of the finite-gap solutions of KdV equation, as well as its hierarchy, subject to periodic boundary conditions. See also Bambusi and Graffi [4], Kappeler and Pöschel [5] for KAM theorems with $d-1 > \delta$. Recently, Liu and Yuan [8] have established a new estimate for the solution of the small-denominators equation with critical unbounded variable coefficients. With the new estimate, a KAM theorem for infinite dimensional Hamiltonian systems including $d-1 > \delta$ and limiting case $d-1 = \delta$ is established in [9], by which quasi-periodic solutions are obtained for a class of derivative nonlinear Schrödinger equations

$$\begin{cases} \mathrm{i}u_t + u_{xx} - M_\sigma u + \mathrm{i}f(u,\bar{u})u_x = 0, \quad (t,x) \in \mathbb{R} \times [0,\pi], \\ u(t,0) = 0 = u(t,\pi), \end{cases} \tag{3}$$

where M_σ is a real Fourier multiplier.

Also by the KAM theorem in [9], Gao and Liu [10] obtained the quasi-periodic solutions for the system

$$\mathrm{i}u_t + u_{xx} + \mathrm{i}f(x,u,\bar{u})u_x + g(x,u,\bar{u}) = 0.$$

By now, not only finite but also infinite Hamiltonian systems have been intensively studied. M.B. Sevryuk ever said in [11] that majority of open problems in KAM theory for finite dimensional Hamiltonian systems could be carried over *mutatis mutandis* to reversible systems. Indeed, reviewing the historical development of the KAM theory, there is a series of remarkable similarities in dynamics between Hamiltonian systems and reversible systems. We refer to [12–15] and references there in. A natural question is that whether or not there is similarities between infinite dimensional Hamiltonian systems and infinite dimensional reversible systems after successfully applying the KAM theory to infinite dimensional Hamiltonian systems coming from PDEs.

Zhang et al. [16] answer the question to some extent. There, they formulate an infinite dimensional KAM theorem for a reversible system with unbounded perturbation by the new estimate in [8], by which quasi-periodic solutions are obtained for the derivative nonlinear Schrödinger equation

$$\begin{cases} \mathrm{i}u_t + u_{xx} + |u_x|^2 u = 0, \quad (t,x) \in \mathbb{R} \times [0,\pi], \\ u(t,0) = 0 = u(t,\pi). \end{cases} \tag{4}$$

However, the classical Schrödinger equation is with potential in physics. To the best of our knowledge, there are few papers studying the existence of time quasi-periodic solutions for (1) when F is an unbounded reversible perturbation. So we will study the derivative nonlinear Schrödinger equation

$$\mathrm{i}u_t = u_{xx} - V(x)u + |u_x|^2 u, \quad (t,x) \in \mathbb{R} \times [0,\pi] \tag{5}$$

subject to Dirichlet boundary conditions

$$u(t,0) = u(t,\pi) = 0, \tag{6}$$

where the potential $V(x)$ is in the square-integrable function space $L^2[0,\pi]$, and the function $F(u,\bar{u},u_x,\bar{u}_x) = |u_x|^2 u$ satisfies

$$F(\bar{u},u,\bar{u}_x,u_x) = \overline{F(u,\bar{u},u_x,\bar{u}_x)}.$$

Consequently, the system is a reversible system with respect to the involution

$$G : (u,\bar{u}) \mapsto (\bar{u},u)$$

on some suitable phase space H^p, such as $H^p = H_0^1([0,\pi])$, the Sobolev space of all complex valued L^2-functions on $[0,\pi]$ with a L^2-derivative and vanishing boundary values.

It is easy to see that $d = 2$ and $\delta = 1$, that is, $d-1 = \delta$ in (5). Moreover, recall that (5) is an infinite dimensional reversible system. Therefore, we would like to apply the KAM theorem in [16] to (5) and get the quasi-periodic solutions for it. Thus, one inevitably encounter the so-called small divisor problem. Some parameters must be introduced in order to adjust frequencies to overcome small divisor problem. Note that (5) does not possess parameters explicitly because of the fixing of $V(x)$. So, now our keypoint lies in introducing parameters into (5).

Refs. [17,18] are the pioneer works extracting parameters from Birkhoff normal forms of Hamiltonian PDEs. In [17], for the nonlinear Schrödinger equation with bounded perturbations, the Birkhoff normal form of order four is used to extract parameters, while in [18], for the nonlinear wave equation, the parameters come from a partial Birkhoff normal form of order four. See also [5] for the Birkhoff normal form of KdV equation, and for example see [20–24] for much more work about Birkhoff normal forms.

Motivated by the method of extracting parameters from a partial Birkhoff normal form in Hamiltonian PDEs, we also reduce the system into a similar normal form to extract parameters. However, the selfadjoint linear operator $A = -d^2/dx^2 + V(x)$ is a classical Sturm–Liouville problem and $|u_x|^2 u$ in (5) is unbounded perturbation with $d-1=\delta$, which will bring two main difficulties. Firstly, by the well-known asymptotics of the spectrum of the Sturm–Liouville problem, the frequencies and corresponding eigenfunctions are

$$\lambda_n = n^2 + c_1 + \frac{c_2}{n^2} + \frac{c_3}{n^3} + O\left(\frac{1}{n^4}\right)$$

and

$$\phi_n(x) = \kappa_n^{-1}\left(\sin nx - \frac{\cos nx}{2n}\int_0^x V(s)\,ds + \tilde{\phi}_n(x)\right),$$

while in [16]

$$\lambda_n = n^2, \qquad \phi_n(x) = \sqrt{\frac{2}{\pi}}\sin nx,$$

they are simple in form clearly. That mainly results in the difficulties in estimating the nonresonant frequencies $|\lambda_l + \lambda_m - \lambda_s - \lambda_j|$ to overcome the small-divisor problem. See Lemma 2.6 for more details. Secondly, there are two parts in the resonant term whose coefficients are W_{lljj} and W_{ljjl} respectively in reducing the system into normal form due to the unboundedness, here $W_{lljj} \neq W_{ljjl}$ while $W_{lljj} = W_{ljjl}$ in the boundedness situation, see [26]. However, in order to apply the KAM theorem we must need the main part in the normal form with the form of $\sum_{l\geqslant 1} l^2 W_{jl}|q_l|^2 q_j$ to guarantee the nondegeneracy, where

$$W_{jl} = \begin{cases} W_{lljj} + \frac{lj}{l^2}W_{ljjl}, & l \neq j, \\ W_{llll}, & l = j. \end{cases}$$

It seems that it is impossible since $W_{ljjl} = ljO(1)$ in form. However, rather fortunately, the coefficient W_{ljjl} equals to $ljO(j^{-1}l^{-1})$ due to the particular structure of $|u_x|^2 u$. Refer to Lemma 2.4 for more details.

After overcoming the above difficulties, we apply the KAM theorem in [16] and have the following results.

Theorem 1. *Arbitrarily fix $n \in \mathbb{N}$. Assume that $V(x)$ is even function in $[-\pi, \pi]$ and analytic in the strip domain $|\operatorname{Im} x| < r$ with $r > 0$. Then we can find many index sets $J = \{j_1 < j_2 < \cdots < j_n\}$ with j_1 large enough to confirm that there exists a Cantor set $\mathcal{C} \subset \mathbb{P}^n = \{I\colon\ I_j > 0 \text{ for } 1 \leqslant j \leqslant n\}$, with positive measure and a family of n-tori*

$$\mathcal{I}_J[\mathcal{C}] = \bigcup_{I\in\mathcal{C}} \mathcal{I}_J(I) \subset E_J = \left\{u = q_1\phi_{j_1} + \cdots + q_n\phi_{j_n}\colon\ q \in \mathbb{C}^n\right\} = \bigcup_{I\in\overline{\mathbb{P}^n}} \mathcal{I}_J(I),$$

given by a Lipschitz continuous embedding

$$\Psi : \mathcal{I}_J[\mathcal{C}] \hookrightarrow H^p,$$

which is a higher order perturbation of the inclusion map $\Psi_0 : E_J \hookrightarrow H^p$ restricted to $\mathcal{I}_J[\mathcal{C}]$. The restriction Ψ of Ψ_0 to each $\mathcal{I}_J(I)$ in the family is an embedding of a rotational n-torus of (5) and it carries quasi-periodic solution of (5).

For one point sets $J = \{j\}$ the same holds except for those $V(x)$ at which

$$\frac{\pm(2 + a_{jj_0})}{1 + a_{jj}} = k = \pm\frac{\lambda_{j_0}}{\lambda_j}$$

with an integer k and some indices $j < j_0 < 3j$, and

$$\frac{\pm(4 + a_{jj_0} + a_{jj_0'})}{1 + a_{jj}} = k = \frac{\pm(\lambda_{j_0} + \lambda_{j_0'})}{\lambda_j}$$

with an integer k and some indices $4j > j_0, j_0' > j$ or $j_0' < j < j_0 < 4j$, where a_{jj}, a_{jj_0} and $a_{jj_0'}$ are defined by (50), and they are uniquely determined by $V(x)$. There are at most finitely many such average values of $V(x)$.

Remark 1. The system (5) is not Hamiltonian, the proof can be found in the Appendix of [16].

Remark 2. Theorem 1 still holds true for

$$F(u, \bar{u}, u_x, \bar{u}_x) = f\left(|u_x|^2\right)u,$$

where f is real analytic in some neighborhood of the origin in $\mathbb{C}$ and satisfies $f(0) = 0$, $f'(0) \neq 0$.

2. The spectra of Sturm–Liouville problems

Consider the Sturm–Liouville (S–L) problems

$$\begin{cases} l(y) := -\dfrac{d^2 y}{dx^2} + V(x)y = \lambda y, \\ y(0) = y(\pi) = 0. \end{cases} \tag{7}$$

It is well known that the S–L problems possess infinite many simple eigenvalues

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \to +\infty. \tag{8}$$

We shall denote by $\phi_n(x)$ the normalized eigenfunctions corresponding to λ_n. Then $\phi_n(x)$ is the solution of the Cauchy problem

$$\begin{cases} -\dfrac{d^2 y}{dx^2} + V(x)y = \lambda_n y, \\ y(0) = 0, \quad y'(0) = \phi_n'(0) \end{cases} \tag{9}$$

on $x \in [0, \pi]$ and satisfies $\phi_n(\pi) = 0$ if $V(x)$ is a smooth function on $[0, \pi]$. In order to extend the eigenfunctions $\phi_n(x)$ from $[0, \pi]$ to odd 2π-periodic analytic functions, we consider the Cauchy problem

$$\begin{cases} -\dfrac{d^2 y}{dx^2} + V(x)y = \lambda y, \\ y(0) = 0, \quad y'(0) = \phi_n'(0) \end{cases} \tag{10}$$

on the interval $[-\pi, \pi]$ under the condition that $V(x)$ is an even and analytic function on $[-\pi, \pi]$. It is easy to prove that the solution $\psi_n(x)$ of (10) is an odd function on $[-\pi, \pi]$ and satisfies:

(i) $\psi_n(x) = \phi_n(x)$ when $x \in [0, \pi]$ by the existence and uniqueness theorem;
(ii) $\psi_n(\pi) = \phi_n(\pi) = \psi_n(0) = \psi_n(-\pi) = 0$.

Therefore $\psi_n(x)$ is the analytic odd continuation of $\phi_n(x)$ on $[-\pi, \pi]$ and is the eigenfunction of the S–L problems

$$\begin{cases} l(y) := -\dfrac{d^2 y}{dx^2} + V(x)y = \lambda y, \\ y(-\pi) = y(\pi) = 0 \end{cases} \tag{11}$$

corresponding to λ_n. In the following we still denote by $\phi_n(x)$ the eigenfunction of (11).

From the argument of [28] we know that the functions ϕ_n's are a normalized orthogonal basis of the space consisting of the square integrable and odd functions in $[-\pi, \pi]$.

Lemma 2.1. *For the eigenvalues λ_n's and eigenfunctions ϕ_n's we have the following asymptotic formulae*

$$\lambda_n = n^2 + c_1 + \frac{c_2}{n^2} + \frac{c_3}{n^3} + O\left(\frac{1}{n^4}\right) \tag{12}$$

and

$$\phi_n(x) = \kappa_n^{-1}\left(\sin nx - \frac{\cos nx}{2n}\int_0^x V(s)\,ds + \tilde{\phi}_n(x)\right) \tag{13}$$

where $\kappa_n > 0$ is a constant depending on n such that $\|\phi_n\|_{L^2_{[-\pi,\pi]}} = 1$, and

$$\tilde{\phi}_n(x) = O\left(\frac{1}{n^2}\right), \qquad \tilde{\phi}_n' = O\left(\frac{1}{n}\right), \qquad \tilde{\phi}_n''(x) = O(1), \qquad ' = \frac{d}{dx}, \tag{14}$$

uniformly for $x \in [-\pi, \pi]$ and

$$c_1 = \frac{1}{\pi}\int_0^{\pi} V(x)\,dx.$$

Proof. The proof can be found in [25] and many textbooks. □

Lemma 2.2. *For κ_n we have the following estimate*:

$$\kappa_n^2 = \pi + O\left(\frac{1}{n^2}\right). \tag{15}$$

Proof. The proof can be found in [19]. □

With respect to $\phi_j(x)$ and $\phi_l'(x)$ we have the following three lemmas.

Lemma 2.3.

$$\kappa_l^2\kappa_j^2\int_{-\pi}^{\pi}\left(\phi_l'(x)\right)^2\phi_j^2(x)\,dx = l^2\begin{cases}\frac{\kappa_j^2}{2} + O(l^{-2}) + O\left(\frac{1}{|l^2-j^2|}\right) + O\left(\frac{|l-j|}{lj(l+j)^2}\right) + O\left(\frac{l+j}{lj(j-l)^2}\right), & j \neq l,\\ \frac{\kappa_j^2}{2} - \frac{\pi}{4} + O(l^{-2}), & j = l.\end{cases} \tag{16}$$

Proof. Set $\mathcal{V}(x) = (1/2)\int_0^x V(s)\,ds$. By (13), we have

$$\phi_n'(x) = \kappa_n^{-1}\left(n\cos nx + \sin nx\,\mathcal{V}(x) - \frac{\cos nx}{2n}V(x) + \tilde{\phi}_n'(x)\right).$$

Then

$$\begin{aligned}\kappa_l^2\kappa_j^2\int_{-\pi}^{\pi}\left(\phi_l'(x)\right)^2\phi_j^2(x)\,dx &= \kappa_j^2\int_{-\pi}^{\pi}\phi_j^2(x)\left(l\cos lx + \sin lx\,\mathcal{V}(x) - \frac{\cos lx}{2l}V(x) + \tilde{\phi}_l'(x)\right)^2 dx\\ &= \kappa_j^2 l^2\left[\frac{1}{2}\int_{-\pi}^{\pi}\phi_j^2(x)\,dx + \int_{-\pi}^{\pi}\phi_j^2\left(\frac{\mathcal{V}(x)}{l}\sin 2lx + \frac{\cos 2lx}{2}\right)dx + O\left(l^{-2}\right)\right]\\ &= \kappa_j^2 l^2\left[\frac{1}{2} + \int_{-\pi}^{\pi}\phi_j^2\left(\frac{\mathcal{V}(x)}{l}\sin 2lx + \frac{\cos 2lx}{2}\right)dx + O\left(l^{-2}\right)\right],\end{aligned} \tag{17}$$

where we have used the fact $\int_{-\pi}^{\pi}\phi_j^2(x)\,dx = 1$.

Then we use the argument similar to that of Lemma 3.2 in [19] we get

$$\kappa_j^2\int_{-\pi}^{\pi}\phi_j^2(x)\cos 2lx\,dx = \int_{-\pi}^{\pi}\left(\sin^2 jx + \frac{1}{j}\sin 2jx\,\mathcal{V}(x)\right)\cos 2lx\,dx + O\left(l^{-2}\right), \tag{18}$$

similarly, we have

$$\frac{\kappa_j^2}{l}\int_{-\pi}^{\pi}\phi_j^2(x)\mathcal{V}(x)\sin 2lx\,dx = \frac{-1}{2l}\int_{-\pi}^{\pi}\cos 2jx\sin 2lx\,\mathcal{V}(x)\,dx + O\left(l^{-2}\right). \tag{19}$$

Observe that

$$\begin{aligned}&\int_{-\pi}^{\pi}\left(\frac{1}{j}\sin 2jx\cos 2lx - \frac{1}{l}\cos 2jx\sin 2lx\right)\mathcal{V}(x)\,dx\\ &\quad = 2\int_0^{\pi}\left(\frac{1}{j}\sin 2jx\cos 2lx - \frac{1}{l}\cos 2jx\sin 2lx\right)\mathcal{V}(x)\,dx\end{aligned}$$

$$= \left(\frac{1}{j} - \frac{1}{l}\right) \int_0^{\pi} \mathcal{V}(x) \sin 2(j+l)x\,dx + \left(\frac{1}{j} + \frac{1}{l}\right) \int_0^{\pi} \mathcal{V}(x) \sin 2(j-l)x\,dx$$

$$= \frac{j-l}{2lj(j+l)}\left[\mathcal{V}(\pi) - \int_0^{\pi} \cos 2(j+l)xV(x)\,dx\right] - \frac{l+j}{2lj(j-l)}\left[\mathcal{V}(\pi) - \int_0^{\pi} \cos 2(j-l)xV(x)\,dx\right]$$

$$= O\left(\frac{1}{|l^2-j^2|}\right) + O\left(\frac{|l-j|}{lj(j+l)^2}\right) + O\left(\frac{l+j}{lj(j-l)^2}\right). \tag{20}$$

Using (17)–(20), we get

$$\kappa_l^2\kappa_j^2 \int_{-\pi}^{\pi} \left(\phi_l'(x)\right)^2 \phi_j^2(x)\,dx$$

$$= l^2\left[\frac{\kappa_j^2}{2} + \frac{1}{2}\int_{-\pi}^{\pi} \sin^2 jx \cos 2lx\,dx + O\left(l^{-2}\right) + O\left(\frac{1}{|l^2-j^2|}\right) + O\left(\frac{|l-j|}{lj(j+l)^2}\right) + O\left(\frac{l+j}{lj(j-l)^2}\right)\right]. \tag{21}$$

Notice that

$$\frac{1}{2}\int_{-\pi}^{\pi} \sin^2 jx \cos 2lx\,dx = \begin{cases} 0, & j \neq l, \\ -\frac{\pi}{4}, & j = l. \end{cases} \tag{22}$$

By (21) and (22), the proof is complete. □

Lemma 2.4. *For $l \neq j$, we have*

$$\kappa_l^2\kappa_j^2 \int_{-\pi}^{\pi} \phi_l'(x)\phi_j'(x)\phi_j(x)\phi_l(x)\,dx = lj\,O\left(j^{-1}l^{-1}\right).$$

Proof. From the process of proving Lemma 2.3, we have

$$\kappa_l^2\kappa_j^2 \int_{-\pi}^{\pi} \phi_l'(x)\phi_j'(x)\phi_j(x)\phi_l(x)\,dx$$

$$= \kappa_j\kappa_l \int_{-\pi}^{\pi} \phi_j(x)\phi_l(x)\left(j\cos jx + \sin jx\,\mathcal{V}(x) - \frac{\cos jx}{2j}V(x) + \tilde{\phi}_j'(x)\right)$$

$$\cdot \left(l\cos lx + \sin lx\,\mathcal{V}(x) - \frac{\cos lx}{2l}V(x) + \tilde{\phi}_l'(x)\right)dx$$

$$= lj\kappa_j\kappa_l\left[\int_{-\pi}^{\pi} \phi_j(x)\phi_l(x)\left(\cos jx + \frac{\sin jx}{j}\mathcal{V}(x) - \frac{\cos jx}{2j^2}V(x) + \frac{1}{j}\tilde{\phi}_j'(x)\right)\right.$$

$$\left.\cdot \left(\cos lx + \frac{\sin lx}{l}\mathcal{V}(x) - \frac{\cos lx}{2l^2}V(x) + \frac{1}{l}\tilde{\phi}_l'(x)\right)dx\right]$$

$$= lj\kappa_j\kappa_l\left[\int_{-\pi}^{\pi} \left(\phi_j(x)\phi_l(x)\cos jx\cos lx + \phi_j(x)\phi_l(x)\cos jx\left(\frac{\sin lx}{l}\mathcal{V}(x) - \frac{\cos lx}{2l^2}V(x) + \frac{1}{l}\tilde{\phi}_l'(x)\right)\right.\right.$$

$$\left.\left. + \phi_j(x)\phi_l(x)\cos lx\left(\frac{\sin jx}{j}\mathcal{V}(x) - \frac{\cos jx}{2j^2}V(x) + \frac{1}{j}\tilde{\phi}_j'(x)\right)\right)dx + O\left(j^{-1}l^{-1}\right)\right].$$

Observe that

$$
\begin{aligned}
&lj\kappa_j\kappa_l \int_{-\pi}^{\pi} \phi_j(x)\phi_l(x)\cos jx\cos lx\,dx \\
&\quad = lj\int_{-\pi}^{\pi}\left(\sin lx - \frac{\cos lx}{l}\mathcal{V}(x) + \tilde{\phi}_l(x)\right)\left(\sin jx - \frac{\cos jx}{j}\mathcal{V}(x) + \tilde{\phi}_j(x)\right)\cos jx\cos lx\,dx \\
&\quad = lj\left[\int_{-\pi}^{\pi}\left(\sin lx\sin jx\cos jx\cos lx + \sin jx\cos jx\cos lx\left(-\frac{\cos lx}{l}\mathcal{V}(x) + \tilde{\phi}_l(x)\right)\right.\right. \\
&\qquad \left.\left. + \sin lx\cos jx\cos lx\left(-\frac{\cos jx}{j}\mathcal{V}(x) + \tilde{\phi}_j(x)\right)\right)dx + O\left(j^{-1}l^{-1}\right)\right].
\end{aligned}
$$

Notice when $l \neq j$,

$$
\int_{-\pi}^{\pi}\sin 2jx\sin 2lx\,dx = \frac{1}{2}\int_{-\pi}^{\pi}\big(-\cos(2j+2l)x + \cos(2j-2l)x\big)\,dx = 0,
$$

then integrating by parts we get

$$
\begin{aligned}
lj\kappa_j\kappa_l \int_{-\pi}^{\pi} \phi_j(x)\phi_l(x)\cos jx\cos lx\,dx &= lj\left[\int_{-\pi}^{\pi}\left(\sin jx\cos jx\cos lx\left(-\frac{\cos lx}{l}\mathcal{V}(x) + \tilde{\phi}_l(x)\right)\right.\right. \\
&\qquad \left.\left. + \sin lx\cos jx\cos lx\left(-\frac{\cos jx}{j}\mathcal{V}(x) + \tilde{\phi}_j(x)\right)\right)dx + O\left(j^{-1}l^{-1}\right)\right] \\
&= lj\left[-\int_{-\pi}^{\pi}\left(\frac{\sin 2jx\cos 2lx}{4l} + \frac{\sin 2lx\cos 2jx}{4j}\right)\mathcal{V}(x)\,dx + O\left(j^{-1}l^{-1}\right)\right] \\
&= lj\left[-\left(\frac{1}{4l} + \frac{1}{4j}\right)\int_{-\pi}^{\pi}\sin(2l+2j)x\,\mathcal{V}(x)\,dx\right. \\
&\qquad \left. + \left(\frac{1}{4l} - \frac{1}{4j}\right)\int_{-\pi}^{\pi}\sin(2l-2j)x\,\mathcal{V}(x)\,dx + O\left(j^{-1}l^{-1}\right)\right] \\
&= lj\,O\left(j^{-1}l^{-1}\right).
\end{aligned}
\tag{23}
$$

Similarly we can prove

$$
\begin{aligned}
&lj\kappa_j\kappa_l \int_{-\pi}^{\pi}\left(\phi_j(x)\phi_l(x)\cos jx\left(\frac{\sin lx}{l}\mathcal{V}(x) - \frac{\cos lx}{2l^2}V(x) + \frac{1}{l}\tilde{\phi}_l'(x)\right)\right. \\
&\quad \left. + \phi_j(x)\phi_l(x)\cos lx\left(\frac{\sin jx}{j}\mathcal{V}(x) - \frac{\cos jx}{2j^2}V(x) + \frac{1}{j}\tilde{\phi}_j'(x)\right)\right)dx = lj\,O\left(j^{-1}l^{-1}\right).
\end{aligned}
\tag{24}
$$

From (23) and (24), the proof is complete. □

Lemma 2.5. *For $j > 5N^2$ and $m, s \in J$, $m \neq s$, we have*

$$
\kappa_j^2\kappa_s\kappa_m \int_{-\pi}^{\pi}\phi_j'(x)\phi_s'(x)\phi_m(x)\phi_j(x)\,dx = js\,O\left(j^{-1}s^{-1}\right).
$$

Proof. From the process of proving Lemma 2.3 and noting that $O(1/s) = O(1/m)$ because of $m, s \in J$, by computation similar to Lemma 2.4 we can obtain the result. □

Lemma 2.6. *Let M be a positive integer large enough and let N be a integer with $N \geqslant M + n - 1$, where n is the integer (mentioned in Theorem 1). For any $l, s, m, j \in \mathbb{N}$ if at least two of them are in $J = \{M, M+1, \ldots, M+n-1\}$, then we have*

$$|\lambda_l + \lambda_m - \lambda_s - \lambda_j| > \frac{\gamma j}{m} \tag{25}$$

unless $\{l, m\} = \{s, j\}$ or $l = j, m \neq s$ with $j \gg 1$. Without loss of generality we assume $j > 5N^2$, where γ is a positive constant depending only on $V(x)$, M and N.

Remark 3. Here we assume $c_2 \neq 0$. This is not essential, for the more precise reason refer to Remark 2 in [19].

Remark 4. Note that there are two plus signs and two minus signs on the left, so the constant c_1 is immaterial.

Proof. Set

$$\Upsilon = \lambda_l + \lambda_m - \lambda_s - \lambda_j.$$

By argument similar to that of Lemma 2.4 in [26], we can have a positive constant $\eta > 0$, such that

$$|\Upsilon| > \eta, \tag{26}$$

where η is a positive constant depending on $V(x)$, M and N.

Case 1. When $j \leqslant 5N^2$, then we have

$$\frac{j}{m} \leqslant 5N^2 < \frac{\eta}{\gamma_1}$$

if we choose a positive constant γ_1 such that $0 < \gamma_1 < \frac{\eta}{5N^2}$. Thus from (26) we have

$$|\Upsilon| > \eta > \gamma_1 \frac{j}{m}.$$

Case 2. $j > 5N^2$.

Case 2.1. $\{l, m\} \cap \{s, j\} \neq \emptyset$.

Case 2.1.1. $l \neq j$, $m = s$ or $l = s$, $m \neq j$. We only prove the first situation, and the second one can be proved similarly. From the assumption of the lemma, we have $\{m, s\} \subset J$, so

$$|\Upsilon| = |\lambda_j - \lambda_l| \geqslant (j + l)|j - l| - j/5 > \frac{4j}{5m}.$$

Case 2.1.2. $l \neq s$, $m = j$. From (26) we have

$$|\Upsilon| > \eta = \eta \cdot \frac{j}{m}.$$

Case 2.1.3. $l = j$, $m \neq s$. This case is excluded.

Case 2.2. $\{l, m\} \cap \{s, j\} = \emptyset$. Note that at least two of $\{l, s, m, j\}$ is in J and $j \notin J$, we have the following subcases.

Case 2.2.1. When $s \geqslant j > l \geqslant m$, then $l, m \in J$. So

$$|\Upsilon| \geqslant |\lambda_j + \lambda_s| - 3N^2 \geqslant j^2 - 3N^2 > j > \frac{j}{m}$$

when $N \gg 1$.

Case 2.2.2. When $j > l \geqslant m > s$, by the argument similar to Case 2.1.1 we have $|\Upsilon| > \frac{2j}{5m}$.

To sum up, let $\gamma = \min\{\gamma_1, \eta, \frac{2}{5}\}$, we complete the proof. □

3. Partial normal form

At first, using a scalar transformation $u = \sqrt{\varepsilon}\tilde{u}$ (ε is a small positive real number) we change (5) into

$$\mathrm{i}\tilde{u}_t = \tilde{u}_{xx} - V(x)\tilde{u} + \varepsilon|\tilde{u}_x|^2\tilde{u}$$

under Dirichlet boundary conditions $\tilde{u}(t,0) = \tilde{u}(t,\pi) = 0$. In the following we omit the '$\sim$' and consider the equation

$$\mathrm{i}u_t = u_{xx} - V(x)u + \varepsilon|u_x|^2u. \tag{27}$$

Now we introduce the Hilbert space $\mathcal{H}^p$: $\bar{q}$ means the complex conjugate of q, and $\mathcal{H}^p$ is the set of all complex sequences $q = (q_j)_{j\geqslant 1}$ satisfying

$$\|q\|_p^2 = \sum_{j\geqslant 1} j^{2p}|q_j|^2 < \infty, \quad p > \frac{3}{2}. \tag{28}$$

It is easy to see that $\mathcal{H}^p$ is a Hilbert space with an inner product corresponding to (28). In fact, $\mathcal{H}^p$ corresponds to the so-called Sobolev space H^p by Fourier transform. Let us introduce some notations. For a vector $q \in \mathcal{H}^p$, we write $q = (q_j\colon\ j \in \mathbb{N})$. Let $\tilde{q} = (q_j\colon\ j \in J)$ and $\hat{q} = (q_j\colon\ j \in \mathbb{N}\setminus J)$. Then $q = \tilde{q} \oplus \hat{q}$. Define $\|\hat{q}\|_p := \|\tilde{q} \oplus \hat{q}\|_p$ by replacing $\tilde{q}$ by 0.

From Section 2, we know that

$$\lambda_n = n^2 + c_1 + \frac{c_2}{n^2} + \frac{c_3}{n^3} + O\left(\frac{1}{n^4}\right)$$

and

$$\phi_n(x) = \kappa_n^{-1}\left(\sin nx - \frac{\cos nx}{2n}\int_0^x V(s)\,ds + \tilde{\phi}_n(x)\right) \quad (n \geqslant 1)$$

are, respectively, eigenvalues and eigenfunctions of the S–L problems.

From now on we focus our attention on the nonlinear $|u_x|^2u$.

Let

$$u(t,x) = \sum_{j\geqslant 1} q_j(t)\phi_j(x)$$

and substitute it into (27). Then we have

$$\dot{q}_j = \mathrm{i}\lambda_j q_j - \mathrm{i}\varepsilon \sum_{l,s,m\geqslant 1} W_{lsmj}\, ls\, q_l\bar{q}_s q_m, \tag{29}$$

where

$$W_{lsmj} = \frac{1}{ls}\int_{-\pi}^{\pi} \phi_l'(x)\phi_s'(x)\phi_m(x)\phi_j(x)\,dx,$$

and (29) is reversible with respect to $G_0 : (q,\bar{q}) \mapsto (\bar{q}, q)$.

To continue our investigation of the system (29) we need to establish the regularity of the nonlinear vector field

$$z = (z_j)_{j\geqslant 1} = \left(\sum_{l,s,m\geqslant 1} W_{lsmj}\, ls\, q_l\bar{q}_s q_m\right)_{j\geqslant 1}.$$

Lemma 3.1. *Suppose that $V(x)$ is real analytic in the strip domain $|\mathrm{Im}\, x| < r$. Then for $p > 3/2$, we have*

$$\|z\|_{p-1} = O\left(\|q\|_p^3\right).$$

The proof of this lemma is given in Appendix A.

Lemma 3.2. *There exists a transformation φ which is bounded in a small neighborhood of the origin in $\mathcal{H}^p$ and changes* (29) *into*

$$\begin{cases} \dot{\check{q}}_j = \mathrm{i}\lambda_j \check{q}_j - \mathrm{i}\varepsilon \sum_{l\geqslant 1} l^2 W_{jl}|\check{q}_l|^2 \check{q}_j - \mathrm{i}\varepsilon g_j, & j \leqslant 5N^2, \\ \dot{\check{q}}_j = \mathrm{i}\lambda_j \check{q}_j - \mathrm{i}\varepsilon \sum_{l\geqslant 1} l^2 W_{jl}|\check{q}_l|^2 \check{q}_j - \mathrm{i}\varepsilon \sum_{\substack{m,s\in J,\\ m\neq s}} jsW_{jsmj}\bar{\check{q}}_s\check{q}_m\check{q}_j - \mathrm{i}\varepsilon g_j, & j > 5N^2, \end{cases} \tag{30}$$

where

$$W_{jl} = \begin{cases} W_{lljj} + \frac{lj}{l^2} W_{ljjl}, & l \neq j, \\ W_{llll}, & l = j, \end{cases}$$

$$\left\| g(\check{q},\bar{\check{q}})\right\|_{p-1} = O\left(\left\|(\check{q}_j)_{j\in\mathbb{N}\setminus J}\right\|_p^3\right) + O\left(\left\|(\check{q}_j)_{j\in\mathbb{N}\setminus J}\right\|_p^2 \left\|(\check{q}_j)_{j\in J}\right\|_p\right) + O\left(\|\check{q}\|_p^5\right)$$

with $g(\check{q},\bar{\check{q}}) = (g_j(\check{q},\bar{\check{q}}))_{j\geqslant 1}$. *Moreover, if* (29) *is reversible with respect to* $G_0 : (q,\bar{q}) \mapsto (\bar{q},q)$, *then* (30) *is also reversible with respect to* $G_0 : (\check{q},\bar{\check{q}}) \mapsto (\bar{\check{q}},\check{q})$.

Proof. We define a change φ in variables:

$$\check{q}_j = q_j + \varepsilon \sum_{l,s,m\geqslant 1} F_{lsmj} q_l \bar{q}_s q_m, \quad j \geqslant 1 \tag{31}$$

with coefficients

$$F_{lsmj} = \begin{cases} \frac{lsW_{lsmj}}{\lambda_l - \lambda_s + \lambda_m - \lambda_j}, & \text{for } (l,s,m,j) \in \mathcal{M}\setminus\mathcal{N}, \\ 0, & \text{otherwise,} \end{cases} \tag{32}$$

where

$$\mathcal{M} = \left\{(l,s,m,j)\in\mathbb{N}^4\colon \text{ at least two of } \{l,s,m,j\} \in J\right\}$$

and

$$\mathcal{N} = \left\{(l,s,m,j)\in\mathcal{M}\colon \{l,m\} = \{s,j\} \text{ or } l=j, m\neq s \text{ for } j > 5N^2\right\}, \quad J = \{M, M+1, \ldots, N\}.$$

By Lemma 2.6, we see that (32) is well defined. Furthermore, we have

$$|F_{lsmj} q_l \bar{q}_s q_m| = \left| \frac{W_{lsmj} l q_l s \bar{q}_s q_m}{\lambda_l - \lambda_s + \lambda_m - \lambda_j} \right| \leqslant \frac{|W_{lsmj}||lq_l||s\bar{q}_s||mq_m|}{\gamma j}. \tag{33}$$

Denote $\breve{q} = (q_1, 2q_2, \ldots) = (jq_j)_{j\geqslant 1}$. Then for $p > 3/2$, by an argument similar to that of Lemma 3.1, we can get

$$\left\| \left(\sum_{l,s,m\geqslant 1} F_{lsmj} q_l \bar{q}_s q_m \right)_{j\geqslant 1} \right\|_p \leqslant C_1 \|\breve{q}\|_{p-1}^3 \leqslant C_2 \|q\|_p^3,$$

where the constant C_2 depends on p and $V(x)$, M and N. Thus the change in variables $\varphi : \mathcal{H}^p \to \mathcal{H}^p$ is analytic in some neighborhood of the origin in $\mathcal{H}^p$. It is easy to see that the change (31) is invertible and its inverse is of the form

$$q_j = \check{q}_j - \varepsilon \sum_{l,s,m\geqslant 1} F_{lsmj} \check{q}_l \bar{\check{q}}_s \check{q}_m + O\left(\|\check{q}\|_p^5\right), \quad j \geqslant 1 \tag{34}$$

in some neighborhood of the origin in $\mathcal{H}^p$. Differentiating both sides of (31) with respect to t, we have

$$\dot{\check{q}}_j = \mathrm{i}\lambda_j q_j - \mathrm{i}\varepsilon \sum_{l,s,m\geqslant 1} W_{lsmj} ls q_l \bar{q}_s q_m + \mathrm{i}\varepsilon \sum_{l,s,m\geqslant 1} F_{lsmj}(\lambda_l - \lambda_s + \lambda_m) q_l \bar{q}_s q_m. \tag{35}$$

Substituting (34) into (35), we have

$$\dot{\check{q}}_j = \mathrm{i}\lambda_j \check{q}_j - \mathrm{i}\varepsilon \sum_{l,s,m\geqslant 1} W_{lsmj} ls \check{q}_l \bar{\check{q}}_s \check{q}_m + \mathrm{i}\varepsilon \sum_{l,s,m\geqslant 1} F_{lsmj}(\lambda_l - \lambda_s + \lambda_m - \lambda_j) \check{q}_l \bar{\check{q}}_s \check{q}_m + O\left(\|\check{q}\|_p^5\right).$$

Using (32), (29) is changed into (30).

Now we prove that the change of variables φ commutes with the involution G_0, i.e. $\varphi \circ G_0 = G_0 \circ \varphi$. Note that W_{lsmj} is in $\mathbb{R}$. Thus by (32), we obtain $F_{lsmj} \in \mathbb{R}$. Then

$$\varphi \circ G_0(q,\bar{q}) = \varphi(\bar{q},q) = (\bar{\check{q}},\check{q}) = G_0(\check{q},\bar{\check{q}}) = G_0 \circ \varphi(q,\bar{q}).$$

This result guarantees that the transformed system (30) is also a reversible system with respect to the involution G_0 : $(\check{q},\bar{\check{q}}) \mapsto (\bar{\check{q}},\check{q})$ by the assumption that (29) is reversible with respect to G_0. $\square$

Remark 5. As the coefficients of transformation (31) are real, so the coefficients of nonnormal terms $-\mathrm{i}g(\check{q},\bar{\check{q}})$ for (30) are still imaginary.

4. Proof of the main theorem

Now from Section 3 we know that the system (5) is reduced into the system

$$\begin{cases} \dot{q}_j = \mathrm{i}\lambda_j q_j - \mathrm{i}\varepsilon \sum_{l\geqslant 1} l^2 W_{jl}|q_l|^2 q_j - \mathrm{i}\varepsilon g_j, & j \leqslant 5N^2, \\ \dot{q}_j = \mathrm{i}\lambda_j q_j - \mathrm{i}\varepsilon \sum_{l\geqslant 1} l^2 W_{jl}|q_l|^2 q_j - \mathrm{i}\varepsilon \sum_{\substack{m,s\in J,\\ m\neq s}} jsW_{jsmj}\bar{q}_s q_m q_j - \mathrm{i}\varepsilon g_j, & j > 5N^2, \end{cases} \tag{36}$$

where

$$W_{jl} = \begin{cases} W_{lljj} + \frac{lj}{l^2} W_{ljjl}, & l \neq j, \\ W_{llll}, & l = j, \end{cases}$$

with

$$\|g(q,\bar{q})\|_{p-1} = O\big(\|(q_j)_{j\in\mathbb{N}\setminus J}\|_p^3\big) + O\big(\|(q_j)_{j\in\mathbb{N}\setminus J}\|_p^2 \|(q_j)_{j\in J}\|_p\big) + O\big(\|q\|_p^5\big).$$

We omit the '$\check{\ }$' for simplicity. By Lemma 3.2, system (36) is still reversible with respect to $G_0 : (q,\bar{q}) \mapsto (\bar{q}, q)$.

We want to apply the KAM theorem in [16] to the system (36). Let $N = M + n - 1$. Rewrite

$$J = \{M, M+1, \dots, N\} := \{j_1, j_2, \dots, j_n\}$$

with $j_1 < j_2 < \cdots < j_n$. For a complex vector $q = (q_j)_{j\geqslant 1} \in \mathcal{H}^p$ and a set J, we write

$$\begin{cases} \tilde{q} = (q_j)_{j\in J}, \\ \hat{q} = (q_j)_{j\in\mathbb{N}\setminus J}, \end{cases} \qquad \begin{cases} Y = \frac{1}{2}\big(|q_j|^2\big)^T_{j\in J}, \\ Z = \frac{1}{2}\big(|q_j|^2\big)^T_{j\in\mathbb{N}\setminus J}. \end{cases}$$

Denote

$$\begin{cases} \beta_1 := (\lambda_j)_{j\in J}, \\ \beta_2 := (\lambda_j)_{j\in\mathbb{N}\setminus J}, \end{cases} \qquad \begin{cases} A := \big(-2W_{jl}l^2\big)_{j,l\in J}, & B_1 := \big(-2W_{jl}l^2\big)_{j\in\mathbb{N}\setminus J,\, l\in J}, \\ B_2 := \big(-2W_{jl}l^2\big)_{j,\, l\in\mathbb{N}\setminus J}, & B_3 := \big(-2W_{jl}l^2\big)_{j\in J,\, l\in\mathbb{N}\setminus J}, \end{cases} \tag{37}$$

then we can rewrite (36) as

$$\begin{cases} \dot{\tilde{q}} = \mathrm{i}\langle\beta_1,\tilde{q}\rangle + \mathrm{i}\varepsilon\big(\mathrm{diag}(AY)\big)\tilde{q} + \mathrm{i}\varepsilon\big(\mathrm{diag}(B_3 Z)\tilde{q} + \mathrm{i}\varepsilon g^1(\tilde{q},\hat{q},\bar{\tilde{q}},\bar{\hat{q}})\big), \\ \dot{\hat{q}} = \mathrm{i}\langle\beta_2,\hat{q}\rangle + \mathrm{i}\varepsilon\big(\mathrm{diag}(B_1 Y)\big)\hat{q} + \mathrm{i}\varepsilon\big(\mathrm{diag}(B_2 Z)\hat{q} + \mathrm{i}\varepsilon\big(\mathrm{diag}\, B_4(\bar{\tilde{q}},\tilde{q})\big)\hat{q} + \mathrm{i}\varepsilon g^2(\tilde{q},\hat{q},\bar{\tilde{q}},\bar{\hat{q}})\big), \end{cases} \tag{38}$$

where

$$B_4(\bar{\tilde{q}},\tilde{q}) = \big(B_4(\bar{\tilde{q}},\tilde{q})_j\big)_{j\in\mathbb{N}\setminus J} := \begin{cases} \sum_{\substack{m,s\in J\\ m\neq s}} jsW_{jsmj}\bar{\tilde{q}}_s\tilde{q}_m, & j > 5N^2, \\ 0, & \text{otherwise}. \end{cases}$$

The system (38) is reversible with respect to the involution $G_0 : (\tilde{q},\hat{q},\bar{\tilde{q}},\bar{\hat{q}}) \mapsto (\bar{\tilde{q}},\bar{\hat{q}},\tilde{q},\hat{q})$ and

$$g^1(\tilde{q},\hat{q},\bar{\tilde{q}},\bar{\hat{q}}) = \big(g^1_j(\tilde{q},\hat{q},\bar{\tilde{q}},\bar{\hat{q}})\big)_{j\in J} = -\big(g_j(\tilde{q},\hat{q},\bar{\tilde{q}},\bar{\hat{q}})\big)_{j\in J},$$
$$g^2(\tilde{q},\hat{q},\bar{\tilde{q}},\bar{\hat{q}}) = \big(g^2_j(\tilde{q},\hat{q},\bar{\tilde{q}},\bar{\hat{q}})\big)_{j\in\mathbb{N}\setminus J} = -\big(g_j(\tilde{q},\hat{q},\bar{\tilde{q}},\bar{\hat{q}})\big)_{j\in\mathbb{N}\setminus J}.$$

Now we introduce a transformation ψ,

$$\begin{cases} \tilde{q}_j = \sqrt{2I_j + 2\xi_j}\, e^{\mathrm{i}\theta_j}, & j \in J, \\ \hat{q}_j = q_j, & j \in \mathbb{N}\setminus J, \end{cases} \tag{39}$$

which transforms (38) into the following form,

$$
\begin{aligned}
&\dot{\theta}_j = \omega_j(\xi) + \varepsilon(AI)_j + \varepsilon(B_3 Z)_j + \frac{\varepsilon}{2}\left(\frac{g_j^1}{\tilde{q}_j} + \frac{\overline{g_j^1}}{\bar{\tilde{q}}_j}\right), \quad j \in J,\\
&\dot{I}_j = \frac{\mathrm{i}\varepsilon}{2}\left(g_j^1 \overline{\tilde{q}_j} - \overline{g_j^1}\tilde{q}_j\right), \quad j \in J,\\
&\dot{q}_j = \mathrm{i}\mu_j(\xi,\theta) q_j + \mathrm{i}\varepsilon(B_1 I)_j q_j + \mathrm{i}\varepsilon\left((B_2 Z)_j q_j + \tilde{g}_j^2\right), \quad j \in \mathbb{N}\setminus J,\\
&\dot{\bar{q}}_j = -\mathrm{i}\mu_j(\xi,\theta)\bar{q}_j - \mathrm{i}\varepsilon(B_1 I)_j \bar{q}_j - \mathrm{i}\varepsilon\left((B_2 Z)_j \bar{q}_j + \overline{\tilde{g}_j^2}\right), \quad j \in \mathbb{N}\setminus J,
\end{aligned}
\tag{40}
$$

where

$$
\begin{aligned}
&\omega_j(\xi) = \lambda_j + \varepsilon(A\xi)_j,\\
&\mu_j(\xi,\theta) = \lambda_j + \varepsilon(B_1\xi)_j + \varepsilon\Omega_j(\xi,\theta),
\end{aligned}
$$

and

$$
\Omega_j(\xi,\theta) = \begin{cases} 2\sum_{\substack{m,s\in J\\ m\neq s}} jsW_{jsmj}\sqrt{\xi_s\xi_m}e^{-\mathrm{i}\theta_s}e^{\mathrm{i}\theta_m}, & j > 5N^2,\\ 0, & \text{otherwise.}\end{cases}
$$

$$
\tilde{g}_j^2 = \sum_{\substack{m,s\in J\\ m\neq s}} jsW_{jsmj}\left[\left(\int_0^1 \sqrt{\frac{\xi_m+\tau I_m}{\xi_s+\tau I_s}}\,d\tau\right) I_s + \left(\int_0^1 \sqrt{\frac{\xi_s+\tau I_s}{\xi_m+\tau I_m}}\,d\tau\right) I_m\right] e^{-\mathrm{i}\theta_s}e^{\mathrm{i}\theta_m} + g_j^2\circ\psi(\theta, I, q, \bar{q}, \xi)
\tag{41}
$$

for $j > 5N^2$, otherwise we have

$$
\tilde{g}_j^2 = g_j^2 \circ \psi(\theta, I, q, \bar{q}, \xi).
$$

Let

$$
\begin{aligned}
&\omega(\xi) := \mathrm{diag}\left\{\left(\omega_j(\xi)\right)_{j\in J}\right\} = \mathrm{diag}(\beta_1 + \varepsilon A\xi),\\
&\Lambda(\xi,\theta) := \mathrm{diag}\left\{\left(\mu_j(\xi,\theta)\right)_{j\in\mathbb{N}\setminus J}\right\} = \mathrm{diag}\left(\beta_2 + \varepsilon B_1\xi + \varepsilon\Omega(\xi,\theta)\right),
\end{aligned}
\tag{42}
$$

then (40) can be rewritten as

$$
\begin{aligned}
&\dot{\theta} = \omega(\xi) + \varepsilon AI + \frac{\varepsilon}{2}B_3|q|^2 + \frac{\varepsilon}{4}\left(\mathrm{diag}\left(\frac{g_j^1}{I_j+\xi_j}\right)\bar{\tilde{q}} + \mathrm{diag}\left(\frac{\overline{g_j^1}}{I_j+\xi_j}\right)\tilde{q}\right),\\
&\dot{I} = \frac{\mathrm{i}\varepsilon}{2}\left(\mathrm{diag}\left(g_j^1\right)\bar{\tilde{q}} - \mathrm{diag}\left(\overline{g_j^1}\right)\tilde{q}\right),\\
&\dot{q} = \mathrm{i}\Lambda(\xi,\theta)q + \mathrm{i}\left(\varepsilon B_1 Iq + \varepsilon B_2 Zq + \varepsilon\tilde{g}^2\right),\\
&\dot{\bar{q}} = -\mathrm{i}\Lambda(\xi,\theta)\bar{q} - \mathrm{i}\left(\varepsilon B_1 I\bar{q} + \varepsilon B_2 Z\bar{q} + \varepsilon\overline{\tilde{g}^2}\right).
\end{aligned}
\tag{43}
$$

It is easy to see that ψ satisfies

$$
G_0\circ\psi = \psi\circ G,
\tag{44}
$$

where $G_0 : (\tilde{q}, \hat{q}, \bar{\tilde{q}}, \bar{\hat{q}}) \mapsto (\bar{\tilde{q}}, \bar{\hat{q}}, \tilde{q}, \hat{q})$ and $G : (\theta, I, q, \bar{q}) \mapsto (-\theta, I, \bar{q}, q)$. Denote by X_{old} and X_{new} the vector fields of (38) and (43) respectively. As (38) is reversible with respect to G_0, thus we have

$$
X_{old}\circ G_0 = -DG_0\cdot X_{old}.
\tag{45}
$$

By the transformation ψ in (39), we get

$$
X_{old}(\psi) = D\psi\cdot X_{new}.
\tag{46}
$$

Moreover by (44), we obtain

$$
D\psi(G)\cdot DG = DG_0(\psi)\cdot D\psi.
\tag{47}
$$

Therefore using (45), (46) and (47), we have

$$
\begin{aligned}
X_{new} \circ G &= (D\psi)^{-1} \cdot X_{old}(\psi \circ G) = (D\psi)^{-1} \cdot X_{old}(G_0 \circ \psi) \\
&= -(D\psi)^{-1}(DG_0) \cdot X_{old}(\psi) \\
&= -DG \cdot (D\psi)^{-1} \cdot X_{old}(\psi) = -DG \cdot X_{new},
\end{aligned}
$$

which implies that (43) is reversible with respect to G.

Now we aim to verify (43) satisfies assumptions (A1)–(A4) in [16]. For convenience, we will adopt lots of notations and definitions from [16].

Lemma 4.1. *When $M \gg 1$, for $n \geqslant 2$ condition* (A1) *is fulfilled.*

Proof. By (42), we obtain

$$
|\omega|_{\Theta}^{lip} = |\beta_1 + \varepsilon A\xi|_{\Theta}^{lip} \leqslant \varepsilon \frac{C^* n N^2}{\pi}, \quad \Theta = \bigl\{\xi \in \mathbb{R}^n \bigm| 0 \leqslant |\xi_j| \leqslant 1\bigr\}, \tag{48}
$$

where C^* is a constant depending only on $V(x)$, and $A = (-2W_{jl}l^2)_{j,l\in J}$, where W_{jl} is given as follows:

$$
W_{jl} = \frac{1}{l^2}\left[\int_{-\pi}^{\pi} \bigl(\phi_l'(x)\bigr)^2 \bigl(\phi_j(x)\bigr)^2\,dx + \int_{-\pi}^{\pi} \phi_l'(x)\phi_j'(x)\phi_j(x)\phi_l(x)\,dx\right].
$$

Noting that $\kappa_n^2 = \pi + O(n^{-2})$, we have $\kappa_n^{-2} = \frac{1}{\pi} + O(n^{-2})$, and from Lemmas 2.3 and 2.4 we have

$$
W_{jl} = \begin{cases} \frac{1}{2\pi} + O(l^{-2}) + O\bigl(\frac{1}{|l^2-j^2|}\bigr) + O\bigl(\frac{|l-j|}{lj(j+l)^2}\bigr) + O\bigl(\frac{l+j}{lj(j-l)^2}\bigr), & j \neq l, \\ \frac{1}{4\pi} + O(l^{-2}), & j = l. \end{cases}
$$

Thus we get

$$
\begin{aligned}
W_{jl} &= \frac{2-\delta_{jl}}{4\pi} + O\bigl(l^{-2}\bigr) + O\bigl(j^{-2}\bigr) + O\left(\frac{1}{|l^2-j^2|}\right), \quad j, l \in J, \\
W_{jl} &= \frac{1}{2\pi} + O\bigl(l^{-2}\bigr) + O\left(\frac{1}{|l^2-j^2|}\right), \quad l \in J,\ j \in \mathbb{N} \setminus J.
\end{aligned} \tag{49}
$$

Let $M_0 = \varepsilon \frac{C^* n N^2}{\pi}$, then we have $|\omega|_{\Theta}^{lip} \leqslant M_0$. Moreover

$$
\det A = (-2)^n j_1^2 j_2^2 \cdots j_n^2 \det(W_{jl})_{j,l\in J}.
$$

Let

$$
a_{jl} = 4\pi W_{jl} - (2 - \delta_{jl}), \tag{50}
$$

from (49) there is a constant b independent of M, such that

$$
|a_{jl}| \leqslant \frac{b}{M}, \quad \text{for } j \in \mathbb{N},\ l \in J.
$$

We have $4\pi(W_{jl})_{j,l\in J} = 2X - E + \tilde{A}$, where E is the identity matrix and all elements of X are 1, and $\tilde{A} = (a_{jl})_{j,l\in J}$. For $\det(2X - E) \neq 0$, by an argument similar to that of Lemma 6.1 in [26] we can prove that $\det(W_{jl})_{j,l\in J} \neq 0$. It follows that $\det A \neq 0$, as a consequence $\omega(\xi)$ is a homeomorphism and Lipschitz continuous in both directions and from (48) there exists $L > 0$ such that

$$
\bigl|\omega^{-1}\bigr|_{\omega(\Theta)}^{lip} \leqslant L.
$$

Clearly, by (12) and (37), $\langle l, \beta_2\rangle \neq 0$ for $1 \leqslant |l| \leqslant 2$. To verify the condition:

$$
\operatorname{meas}\bigl(\bigl\{\xi\colon \bigl\langle k, \omega(\xi)\bigr\rangle + \bigl\langle l, \Lambda(\xi)\bigr\rangle = 0\bigr\}\bigr) = 0
$$

we have to check that

$$
\langle \beta_1, k\rangle + \langle \beta_2, l\rangle \neq 0 \quad \text{or} \quad A^T k + B_1^T l \neq 0
$$

for all $(k, l) \in \mathbb{Z}^n \times \mathbb{Z}^\infty$ with $1 \leqslant |l| \leqslant 2$.

Suppose

$$A^T k + B_1^T l = 0, \tag{51}$$

and multiply the matrix

$$D = \operatorname{diag}\left(\frac{-1}{2j_1^2}, \frac{-1}{2j_2^2}, \dots, \frac{-1}{2j_n^2}\right)$$

from the left-hand side of (51) and then we obtain an equivalent form

$$(W_{jl})_{j,l\in J}^T k + (W_{jl})_{j\in\mathbb{N}\setminus J, l\in J} l := A_1 k + B^T l = 0. \tag{52}$$

Write l as

$$l = (\dots, l_j, \dots, l_{j_0}, \dots, l_{j_0'}, \dots),$$

where $l_j = 0$ for $j \notin \{j_0, j_0'\}$ and l_{j_0}, $l_{j_0'} \in \{0, 1, -1\}$ with $|l_{j_0}| + |l_{j_0'}| \neq 0$.

Let $k = (k_1, \dots, k_s, \dots, k_n)$, multiplying both sides of Eq. (52) by 4π from (49) we have

$$\sum_{t=1}^n (2 + a_{i_s i_t}) k_t - k_s + (2 + a_{i_s j_0}) l_{j_0} + (2 + a_{i_s j_0'}) l_{j_0'} = 0, \quad s = 1, 2, \dots, n. \tag{53}$$

Taking $s = s_1$ and $s = s_2$ in (53), we get two equations (53-s_1) and (53-s_2). Considering (53-s_1) minus (53-s_2), we get

$$\sum_{t=1}^n (a_{i_{s_1} i_t} - a_{i_{s_2} i_t}) k_t + k_{s_2} - k_{s_1} + (a_{i_{s_1} j_0} - a_{i_{s_2} j_0}) l_{j_0} + (a_{i_{s_1} j_0'} - a_{i_{s_2} j_0'}) l_{j_0'} = 0. \tag{54}$$

Moreover, by (53) and (54) and noting $i_s, i_t \in J$, we have

$$\begin{aligned} |k_{s_2} - k_{s_1}| &\leqslant \left|\sum_{t=1}^n (a_{i_{s_1} i_t} - a_{i_{s_2} i_t}) k_t\right| + |a_{i_{s_1} j_0} - a_{i_{s_2} j_0}| + |a_{i_{s_1} j_0'} - a_{i_{s_2} j_0'}| \\ &\leqslant \sum_{t=1}^n |a_{i_{s_1} i_t} - a_{i_{s_2} i_t}||k_t| + |a_{i_{s_1} j_0} - a_{i_{s_2} j_0}| + |a_{i_{s_1} j_0'} - a_{i_{s_2} j_0'}| \\ &< \frac{2b}{M} \sum_{t=1}^n |k_t| + \frac{4b}{M}. \end{aligned} \tag{55}$$

Assume $|k_{s_2} - k_{s_1}| < 1$ (in the following we will prove the inequality), then $k_{s_2} = k_{s_1}$, since k is an integer vector. Thus there is a integer h such that all elements of k are equal to h. Therefore we can rewrite Eqs. (53) as

$$(2n - 1)h + h \sum_{t=1}^n a_{i_s i_t} + (2 + a_{i_s j_0}) l_{j_0} + (2 + a_{i_s j_0'}) l_{j_0'} = 0, \quad s = 1, 2, \dots, n. \tag{56}$$

From (56) we get

$$(2n - 1)h + 2(l_{j_0} + l_{j_0'}) = -h \sum_{t=1}^n a_{i_s i_t} - a_{i_s j_0} l_{j_0} - a_{i_s j_0'} l_{j_0'}. \tag{57}$$

If (57) admits a nonzero integer solution h_0, by $i_s, i_t \in J$, we can choose M large enough, such that

$$\left| h_0 \sum_{t=1}^n a_{i_s i_t}\right| + |a_{i_s j_0}| + |a_{i_s j_0'}| < \frac{h_0 n b}{M} + \frac{2b}{M} < 1,$$

then $(2n - 1)h_0 + 2(l_{j_0} + l_{j_0'}) = 0$. It is evident that for $n \geqslant 2$, the only integer solution to this equation is $h_0 = 0$ and $l_{j_0} + l_{j_0'} = 0$, this is contrary to our assumption. So we have no nonzero integer h satisfying (56). Consequently, h has to be 0.

It follows from $h = 0$ that $k = 0$. Therefore

$$\langle \beta_1, k\rangle + \langle \beta_2, l\rangle = \langle \beta_2, l\rangle \neq 0.$$

Then we complete the proof of the measure condition.

Finally let us verify the assumption $|k_{s_2}-k_{s_1}|<1$. To this end, we need to estimate $\sum_{t=1}^{n}|k_t|$. With (53), we have

$$\begin{aligned}\sum_{s=1}^{n}|k_s| &= \sum_{s=1}^{n}\left|\sum_{t=1}^{n}(2+a_{i_s i_t})k_t+(2+a_{i_s j_0})l_{j_0}+(2+a_{i_s j_0'})l_{j_0'}\right| \\ &\leqslant \sum_{s=1}^{n}\left(\left|\sum_{t=1}^{n}2k_t\right|+\sum_{t=1}^{n}|a_{i_s i_t}k_t|+|a_{i_s j_0}|+|a_{i_s j_0'}|+4\right) \\ &< 2n\left|\sum_{t=1}^{n}k_t\right|+\frac{nb}{M}\sum_{t=1}^{n}|k_t|+\left(4+\frac{2b}{M}\right)n.\end{aligned}$$

Summing up Eqs. (53) from $s=1$ to n, we have

$$(2n-1)\sum_{s=1}^{n}k_s+\sum_{s=1}^{n}\sum_{t=1}^{n}a_{i_s i_t}k_t+\sum_{s=1}^{n}(2+a_{i_s j_0})l_{j_0}+\sum_{s=1}^{n}(2+a_{i_s j_0'})l_{j_0'}=0,$$

consequently we get

$$\begin{aligned}(2n-1)\left|\sum_{s=1}^{n}k_s\right| &\leqslant \left|\sum_{s=1}^{n}\sum_{t=1}^{n}a_{i_s i_t}k_t\right|+\left|\sum_{s=1}^{n}(2+a_{i_s j_0})l_{j_0}\right|+\left|\sum_{s=1}^{n}(2+a_{i_s j_0'})l_{j_0'}\right| \\ &\leqslant \sum_{s=1}^{n}\sum_{t=1}^{n}|a_{i_s i_t}k_t|+\left(4+\frac{2b}{M}\right)n \\ &\leqslant \frac{nb}{M}\sum_{t=1}^{n}|k_t|+\left(4+\frac{2b}{M}\right)n.\end{aligned}$$

Thus

$$\sum_{s=1}^{n}|k_s|<\frac{4n-1}{2n-1}\cdot\frac{nb}{M}\sum_{t=1}^{n}|k_t|+\frac{4n-1}{2n-1}\left(4+\frac{2b}{M}\right)n,$$

therefore

$$\sum_{t=1}^{n}|k_t|<\frac{(4+\frac{2b}{M})n}{\frac{2n-1}{4n-1}-\frac{nb}{M}}<\frac{(12+\frac{2b}{M})n}{M-3nb}M.$$

If M satisfies $M-3nb>100nb$, then $\frac{b}{M}<\frac{1}{103n}<\frac{1}{103}$, so $\sum_{t=1}^{n}|k_t|<\frac{M}{4b}$. It follows from (55) that

$$|k_{s_2}-k_{s_1}|<\frac{2b}{M}\cdot\frac{M}{4b}+\frac{4b}{M}<\frac{1}{2}+\frac{1}{2}<1$$

for $M\gg 1$.

In order to complete the proof of the lemma we have to verify $\langle l,\Lambda(\xi)\rangle\neq 0$. It is evident that $\langle l,\Lambda(\xi)\rangle$ does not vanish for small $|\xi|$ because of the asymptotic behavior of the frequencies and the assumption $\langle l,\beta_2\rangle\neq 0$. So far condition (A1) is satisfied. □

Lemma 4.2. *For $J=\{j\}$ and $j\gg 1$*

$$\operatorname{meas}\bigl(\bigl\{\xi\colon \langle k,\omega(\xi)\rangle+\langle l,\Lambda(\xi)\rangle=0\bigr\}\bigr)=0 \tag{58}$$

holds for all k,l with $1\leqslant|l|\leqslant 2$ and all analytic functions $V(x)$ in some complex strip domain except those at which

$$\frac{\pm(2+a_{jj_0})}{1+a_{jj}}=k=\pm\frac{\lambda_{j_0}}{\lambda_j}$$

with an integer k and some indices $j<j_0<3j$, and

$$\frac{\pm(4+a_{jj_0}+a_{jj_0'})}{1+a_{jj}}=k=\frac{\pm(\lambda_{j_0}+\lambda_{j_0'})}{\lambda_j}$$

with an integer k and some indices $4j>j_0$, $j_0'>j$ or $j_0'<j<j_0<4j$. There are at most finitely many such average values of $V(x)$.

Proof. We continue the preceding proof. Suppose to the contrary that $\langle\beta_1, k\rangle = \langle\beta_2, l\rangle$ and $A_1 k = B_1^T l$. For $J = \{j\}$, from Eq. (53) we get that

$$k = \frac{(2 + a_{jj_0}) l_{j_0} + (2 + a_{jj'_0}) l_{j'_0}}{1 + a_{jj}}. \tag{59}$$

The assumption $\langle\beta_1, k\rangle = \langle\beta_2, l\rangle$ then further implies

$$k = \frac{\lambda_{j_0} l_{j_0} + \lambda_{j'_0} l_{j'_0}}{\lambda_j}. \tag{60}$$

Case 1. When $|l| = 1$, without loss of generality we assume $l_{j_0} = \pm 1$, $l_{j'_0} = 0$, then (59) and (60) are reduced to

$$k = \frac{\pm(2 + a_{jj_0})}{1 + a_{jj}} \tag{61}$$

and

$$k = \frac{\pm\lambda_{j_0}}{\lambda_j} \tag{62}$$

respectively. Note that when $j \gg 1$, we have

$$|a_{jj}| + |a_{jj_0}| < \frac{2b}{M} < \frac{1}{2},$$

thus from (61) we get $1 < k < 5$, together with (62) we get $j < j_0 < 3j$.

Case 2. When $|l| = 2$, we divide this situation into two cases to consider.

Case 2.1. $l_{j_0} = -l_{j'_0} \neq 0$. Since (59) is reduced to

$$k = \frac{a_{jj_0} - a_{jj'_0}}{1 + a_{jj}},$$

we can get $k = 0$ when $j \gg 1$. Thus for $n \geqslant 1$ we have $\langle\beta_1, k\rangle \neq \langle\beta_2, l\rangle$ for all $k \in \mathbb{Z}$ and $l \in \mathbb{Z}^\infty$ with $1 \leqslant |l| \leqslant 2$ and all $V(x)$.

Case 2.2. When $l_{j_0} = l_{j'_0} = \pm 1$, without loss of generality we assume $j'_0 < j_0$, (59) and (60) are reduced to

$$k = \frac{\pm(4 + a_{jj_0} + a_{jj'_0})}{1 + a_{jj}} \tag{63}$$

and

$$k = \frac{\pm(\lambda_{j_0} + \lambda_{j'_0})}{\lambda_j}. \tag{64}$$

Also we have

$$|a_{jj}| + |a_{jj_0}| + |a_{jj'_0}| < \frac{3b}{M} < \frac{1}{2}$$

when $j \gg 1$. Thus from (63) we have $2 < k < 10$. Together with (64) we can get

$$j < j'_0 < j_0 < 4j \quad \text{or} \quad j'_0 < j < j_0 < 4j.$$

One then verifies that for $j_0 \geqslant 4j$ there are no solutions at all. Hence there are at most finitely many exceptional average values of $V(x)$ for each j. $\square$

Lemma 4.3. *$|\mu_j(\xi, \theta) - \mu_l(\xi, \theta)| \geqslant c_0 |j - l|(j + l)$ for any $j, l \geqslant 1$ and $\xi \in \Theta$, $\theta \in \{\theta \in \mathbb{C}^n \mid |\mathrm{Im}\,\theta| < 1\}$, where c_0 is a positive constant depending on $V(x)$ and M.*

Proof. Recall that $\mu_j(\xi,\theta)=\lambda_j+\varepsilon(B_1\xi)_j+\varepsilon\Omega_j(\xi,\theta)$. Since $\varepsilon\Omega_j(\xi,\theta)=\varepsilon O(1)$ from Lemma 2.5 and for any $j,l\geqslant 1$,

$$\begin{aligned}(B_1\xi)_j-(B_1\xi)_l&=\sum_{i=j_1}^{j_n}\left(O\left(i^{-2}\right)+O\left(\frac{1}{|j^2-i^2|}\right)+O\left(\frac{1}{|l^2-i^2|}\right)\right)i^2\\&=O(1)+\sum_{i=j_1}^{j_n}\left(O\left(\frac{i^2}{|j^2-i^2|}\right)+O\left(\frac{i^2}{|l^2-i^2|}\right)\right),\end{aligned}$$

thus for $j\neq l$, without loss of generality, we assume $1\leqslant l<j$, then we have

$$\begin{aligned}\left|\mu_j(\xi,\theta)-\mu_l(\xi,\theta)\right|&=\left|\lambda_j-\lambda_l+\varepsilon\left((B_1\xi)_j-(B_1\xi)_l\right)+\varepsilon\left(\Omega_j(\xi,\theta)-\Omega_l(\xi,\theta)\right)\right|\\&=\left|j^2-l^2+c_2\left(j^{-2}-l^{-2}\right)+O\left(l^{-3}\right)+\varepsilon\left(O(1)\right)+\sum_{i=j_1}^{j_n}\left(O\left(\frac{i^2}{|j^2-i^2|}\right)+O\left(\frac{i^2}{|l^2-i^2|}\right)\right)\right|\\&=\left(j^2-l^2\right)\left|1+O\left(\frac{1}{j^2l^2}\right)+\varepsilon\left(O\left(\frac{1}{j^2-l^2}\right)+\sum_{i=j_1}^{j_n}\left(O\left(\frac{i^2}{(j^2-l^2)|j^2-i^2|}\right)\right.\right.\right.\\&\qquad\left.\left.\left.+O\left(\frac{i^2}{(j^2-l^2)|l^2-i^2|}\right)\right)\right)\right|.\end{aligned}\tag{65}$$

Note $j,l\notin J$, then we must analyze two different cases according to whether $j\leqslant M$ or not.

Case 1. When $1\leqslant l<j\leqslant M$, let

$$\tilde{c}_0:=\inf_{1\leqslant l<j\leqslant M}\left|1+O\left(\frac{1}{j^2l^2}\right)+\varepsilon\left(O\left(\frac{1}{j^2-l^2}\right)+\sum_{i=j_1}^{j_n}\left(O\left(\frac{i^2}{(j^2-l^2)|j^2-i^2|}\right)+O\left(\frac{i^2}{(j^2-l^2)|l^2-i^2|}\right)\right)\right)\right|>0,$$

thus we have $|\mu_j(\xi,\theta)-\mu_l(\xi,\theta)|\geqslant\tilde{c}_0|j-l|(j+l)$.

Case 2. When $1\leqslant l<M<N<j$, we have

$$\begin{aligned}\left|\mu_j(\xi,\theta)-\mu_l(\xi,\theta)\right|&=(j-l)(j+l)\left|1+O\left(\frac{1}{j^2l^2}\right)+\varepsilon\left(O\left(\frac{1}{j^2-l^2}\right)+O\left(\frac{1}{(j^2-l^2)(1-l^2/j_n^2)}\right)\right)\right|\\&\geqslant\tilde{c}_1(j-l)(j+l),\end{aligned}$$

since

$$\left|1+O\left(\frac{1}{j^2l^2}\right)+\varepsilon\left(O\left(\frac{1}{j^2-l^2}\right)+O\left(\frac{1}{(j^2-l^2)(1-l^2/j_n^2)}\right)\right)\right|\geqslant 1-\frac{\tilde{c}_2}{M^2}-\frac{\tilde{c}_3\varepsilon}{M}>\tilde{c}_1>0$$

for $M\gg 1$.

Similarly we can get

$$\left|\mu_j(\xi,\theta)-\mu_l(\xi,\theta)\right|\geqslant\tilde{c}_4(j-l)(j+l)$$

when $N<l<j$.

Defining $c_0:=\min\{\tilde{c}_0,\tilde{c}_1,\tilde{c}_4\}$, we have

$$\left|\mu_j(\xi,\theta)-\mu_l(\xi,\theta)\right|\geqslant c_0|j-l|(j+l)$$

and c_0 is a positive constant depending on $V(x)$ and M. □

Moreover since

$$\varepsilon\left|\Omega_j(\xi,\theta)\right|_{\Theta}^{lip}\leqslant\varepsilon\tilde{C}_1,\qquad\varepsilon\left|(B_1\xi)_j\right|_{\Theta}^{lip}\leqslant\varepsilon\tilde{C}_2nN^2,$$

thus we have

$$\left|\Lambda(\xi,\theta)\right|_{-\delta,\Theta}^{lip}=\sup_{j\geqslant 1}j^{-\delta}\left|\varepsilon\mu_j(\xi,\theta)\right|_{\Theta}^{lip}\leqslant\varepsilon\tilde{C}nN^2,$$

where $\tilde{C}_1, \tilde{C}_2$ and $\tilde{C}$ are constants depending only on $V(x), \delta = d-1, d=2$. Let $C_\lambda^{lip} = \varepsilon \tilde{C} n N^2$, then $|\Lambda(\xi,\theta)|_{-\delta,\Theta}^{lip} \leqslant C_\lambda^{lip}$. And if ε is small enough, $64LC_\lambda^{lip} \leqslant c_0$ is true.

Thus the system (40) satisfies condition (A2).

It is sufficient to verify the regularity of $f = (f_1, f_2, \mathrm{i} f_3, -\mathrm{i}\overline{f_3})^T$ in (40), where

$$\begin{aligned} f_1 &= \varepsilon A I + \frac{\varepsilon}{2} B_3 |q|^2 + \frac{\varepsilon}{4}\left(\operatorname{diag}\left(\frac{g_j^1}{I_j + \xi_j}\right)\bar{\tilde{q}}(I,\theta) + \operatorname{diag}\left(\frac{\overline{g_j^1}}{I_j+\xi_j}\right)\tilde{q}(I,\theta)\right), \\ f_2 &= \frac{\mathrm{i}\varepsilon}{2}\left(\operatorname{diag}\left(g_j^1\right)\bar{\tilde{q}}(I,\theta) - \operatorname{diag}\left(\overline{g_j^1}\right)\tilde{q}(I,\theta)\right), \\ f_3 &= \varepsilon B_1 I q + \varepsilon B_2 Z q + \varepsilon \tilde{g}^2, \\ \overline{f_3} &= \varepsilon B_1 I \bar{q} + \varepsilon B_2 Z \bar{q} + \varepsilon \overline{\tilde{g}^2}. \end{aligned}$$

From Lemma 3.2 and (41), we have

$$\begin{aligned} \left\| g^1(\tilde{q}, \bar{\tilde{q}}, q, \bar{q},)\right\| &= O\left(\|q\|_p^3\right) + O\left(\|q\|_p^2 \|\tilde{q}\|\right) + O\left(\|\tilde{q} \oplus q\|_p^5\right), \\ \left\|\tilde{g}^2\right\|_{p-1} &= O(I) + O\left(\|q\|_p^3\right) + O\left(\|q\|_p^2\|\tilde{q}\|\right) + O\left(\|\tilde{q}\oplus q\|_p^5\right). \end{aligned}$$

Now let $0 < r < 1/2$ and consider the phase space domain

$$\mathcal{D}(1,r) := \left\{|\operatorname{Im}\theta| < 1,\ |I| < r^2,\ \|q\|_p < r,\ \|\bar{q}\|_p < r\right\},$$

set

$$\tilde{\Theta} := \left\{\xi \in \mathbb{R}^n \colon\ \frac{r}{2} < |\xi| < r\right\}.$$

We obtain that the perturbation f is analytic on $\mathcal{D}(1,r) \times \tilde{\Theta}$.

Now let ε be sufficiently small, f satisfies the smallness condition. It is easy to verify that the other conditions and assumptions of Theorem 1.1 in [16] are satisfied, too. So using the KAM theorem [16], there exist a Cantor set $\Pi \subset \tilde{\Theta}$, a Lipschitz family of smooth torus embeddings $\Psi : \mathbb{T}^n \times \Pi \to \mathcal{P}^p$, and a Lipschitz map $\omega^* : \Pi \to \mathbb{R}^n$, such that for each $\xi \in \Pi$ the map Ψ restricted to $\mathbb{T}^n \times \{\xi\}$ is a smooth embedding of a rotational torus with frequencies $\omega^*(\xi)$, in other words,

$$t \mapsto \Psi\left(\theta + t\omega^*(\xi), \xi\right), \quad t \in \mathbb{R}$$

is a smooth quasi-periodic solution, for every $\theta \in \mathbb{T}^n$ and $\xi \in \Pi$. Furthermore, the coordinate transformations φ and ψ are homeomorphisms. So $\varphi^{-1}(\psi \circ \Psi(\theta + t\omega^*(\xi), \xi))$ is a quasi-periodic solution of (29). Going back to the Sobolev space H^p by the isometry $\mathcal{S}$,

$$\mathcal{S} : \mathcal{H}^p \to H^p, \qquad q \mapsto \mathcal{S}q = \sum_{j \geqslant 1} q_j(t)\phi_j(x),$$

we obtain the time quasi-periodic solution $u = \mathcal{S}\varphi^{-1}(\psi \circ \Psi(\theta + t\omega^*(\xi), \xi))$ of Eq. (5).

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Appendix A. A direct proof of Lemma 3.1

Suppose that the potential $V(x)$ is analytic in $|\operatorname{Im} x| < r$, then the eigenfunctions are analytic in $|\operatorname{Im} x| < r$. If we let

$$\phi_i(x) = \sum_{n \in \mathbb{Z}} a_i^n e^{\mathrm{i}\langle n, x\rangle},$$

then (see, e.g., [27]) we have

$$\left|a_i^n\right| < c e^{-|i-n| r},$$

where c is a constant depending on $V(x)$ and r. Later, the letter c denotes suitable (possibly different) constants.

Recall that $z_j = \sum_{l,s,m\geqslant 1} W_{lsmj} lsq_l\bar{q}_s q_m$, where

$$\begin{aligned}
W_{lsmj}ls &= \int_{-\pi}^{\pi} \phi_l'(x)\phi_s'(x)\phi_m(x)\phi_j(x)\,dx \\
&= \int_{-\pi}^{\pi} \left(\sum_{n_0\in\mathbb{Z}} a_l^{n_0} e^{\mathrm{i}\langle n_0,x\rangle}\right)'\left(\sum_{n_1\in\mathbb{Z}} a_l^{n_1} e^{\mathrm{i}\langle n_1,x\rangle}\right)'\left(\sum_{n_2\in\mathbb{Z}} a_l^{n_2} e^{\mathrm{i}\langle n_2,x\rangle}\right)\left(\sum_{n_3\in\mathbb{Z}} a_l^{n_3} e^{\mathrm{i}\langle n_3,x\rangle}\right) dx \\
&= \int_{-\pi}^{\pi} \left(\sum_{n_0\in\mathbb{Z}} \mathrm{i}n_0 a_l^{n_0} e^{\mathrm{i}\langle n_0,x\rangle}\right)\left(\sum_{n_1\in\mathbb{Z}} \mathrm{i}n_1 a_l^{n_1} e^{\mathrm{i}\langle n_1,x\rangle}\right)\left(\sum_{n_2\in\mathbb{Z}} a_l^{n_2} e^{\mathrm{i}\langle n_2,x\rangle}\right)\left(\sum_{n_3\in\mathbb{Z}} a_l^{n_3} e^{\mathrm{i}\langle n_3,x\rangle}\right) \\
&= -\sum_{n_0+n_1+n_2+n_3=0} n_0 n_1 a_l^{n_0} a_s^{n_1} a_m^{n_2} a_j^{n_3},
\end{aligned}$$

it follows that

$$\begin{aligned}
\|z\|_{p-1}^2 &= \sum_{j\geqslant 1} |z_j|^2 j^{2(p-1)} \leqslant \left(\sum_{j\geqslant 1} |z_j| j^{p-1}\right)^2 \\
&= \left(\sum_{j\geqslant 1}\left| -\sum_{l,s,m\geqslant 1}\sum_{n_0+n_1+n_2+n_3=0} n_0 n_1 a_l^{n_0} a_s^{n_1} a_m^{n_2} a_j^{n_3} q_l\bar{q}_s q_m\right| j^{p-1}\right)^2 \\
&\leqslant c\left(\sum_{j,l,s,m\geqslant 1}\sum_{n_0+\cdots+n_3=0} |n_0||n_1|\left|a_l^{n_0}\right|\left|a_s^{n_1}\right|\left|a_m^{n_2}\right|\left|a_j^{n_3}\right||q_l||\bar{q}_s||q_m| j^{p-1}\right)^2 \\
&\leqslant c\left(\sum_{l,s,m\geqslant 1}\sum_{n_0+\cdots+n_3=0}\sum_{j\geqslant 1} e^{-|n_3-j|r} j^{p-1}|n_0||n_1|\left|a_l^{n_0}\right|\left|a_s^{n_1}\right|\left|a_m^{n_2}\right||q_l||\bar{q}_s||q_m|\right)^2. \qquad (66)
\end{aligned}$$

Note that the inequalities

$$\sum_{j\in\mathbb{Z}} e^{\frac{-|n-j|r}{2}}|j|^a \leqslant c\sum_{j\in\mathbb{Z}}\big(1+|n-j|\big)^{-K}|j|^a$$

and

$$\sum_{j\in\mathbb{Z}}\big(1+|n-j|\big)^{-K}|j|^a \leqslant c\int_{\mathbb{R}}\big(1+|n-x|\big)^{-K}x^a\,dx \leqslant c|n|^a$$

hold true for $K > a+1$. Thus by the Cauchy inequality we have

$$\begin{aligned}
(66) &\leqslant c\left(\sum_{l,s,m\geqslant 1}\sum_{n_0+\cdots+n_3=0} |n_3|^{p-1}|n_0||n_1|\left|a_l^{n_0}\right|\left|a_s^{n_1}\right|\left|a_m^{n_2}\right||q_l||\bar{q}_s||q_m|\right)^2 \\
&\leqslant c\left(\sum_{l,s,m\geqslant 1}\sum_{n_0,n_1,n_2\in\mathbb{Z}} |n_0+n_1+n_2|^{p-1}|n_0||n_1| e^{-|n_0-l|r}e^{-|n_1-s|r}e^{-|n_2-m|r}|q_l||\bar{q}_s||q_m|\right)^2 \\
&\leqslant c\left(\sum_{l,s,m\geqslant 1}\sum_{n_0,n_1,n_2\in\mathbb{Z}} |n_0+n_1+n_2|^{p-1} e^{-|n_0-l|r/2}e^{-|n_1-s|r/2}e^{-|n_2-m|r}|q_l||\bar{q}_s||q_m|\right)^2 \\
&\leqslant c\left(\sum_{l,s,m\geqslant 1} |l|^{p-1}|s|^{p-1}|m|^{p-1}|q_l||\bar{q}_s||q_m|\right)^2 \\
&\leqslant c\left(\sum_{l\geqslant 1}|l|^{2p}|q_l|^2\cdot\sum_{l\geqslant 1}|l|^{-2}\right)^3 = c\|q\|_p^6.
\end{aligned}$$

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